

1 Galilean Relativity

1.1. RELATIVITY AND COMMON SENSE

A child walks along the floor of a moving train. Passengers on the train measure the child's speed and find it to be 1 meter per second. When ground-based observers measure the speed of the same child, they obtain a different value; observers on an airplane flying overhead obtain still another. Each set of observers obtains a different value when measuring the same physical quantity. Finding the relation between those values is a typical problem in relativity.

There is nothing at all startling about these observations; relativity was not invented by Albert Einstein. Einstein's work did, however, drastically change the way such phenomena are understood; the term "relativity" as used today generally refers to Einstein's theory.

The study of relativity began with the work of Galileo Galilei around 1630; Isaac Newton also made important contributions. The ideas described in this chapter, universally accepted until 1900, are known as "Galilean relativity."

Galilean relativity is fully consistent with the intuitive notions that we call "common sense."¹ In the example above, if the train moves at 30 meters per second (m/sec) in the same direction as the child, common sense suggests that ground-based observers should find the child's speed to be 31 m/sec; Galilean relativity gives precisely that value. Einstein's theory, as we shall see, gives a different result.

In the case of the child, the difference between the two theories is minute. The speed measured by ground observers according to Einstein's

1. According to Einstein, common sense is "that layer of prejudice laid down in the mind prior to the age of eighteen."

relativity differs from the Galilean value 31 m/sec only in the fourteenth decimal place; no measurement could possibly detect such a tiny difference. This result is characteristic of Einsteinian relativity: its predictions are indistinguishable from those of Galilean relativity whenever the observers, as well as all objects under observation, move slowly relative to one another. That realm is generally called the nonrelativistic limit, although Galilean or Newtonian limit would be a more apt designation. “Slowly” here means at a speed much less than the speed of light.

The speed of light plays a central role in Einstein’s theory; whenever any speed in the problem approaches that value, Einsteinian relativity departs dramatically from that of Galileo and Newton. Because the speed of light is so great, however, most commonly observed phenomena are adequately described by Galilean relativity.

The “special” theory of relativity, which is the principal subject of this book, is restricted to observers who move *uniformly*, that is, at constant speed in the same direction. If observers move with changing speeds, or along curved paths, the problem of relating their measurements is much more complicated. Einstein addressed that problem as well, in his “general” theory of relativity. Because the general theory involves quite advanced mathematics, I can give only a descriptive treatment in chapter 8. The special theory, in contrast, requires only elementary algebra and geometry and can be presented with full rigor.

Many of the conclusions of special relativity run counter to our intuition concerning the nature of space and time. Before Einstein, no one doubted that time is absolute. Newton put it as follows in his *Principia*: “Absolute, true, and mathematical time, of itself and from its own nature, flows equably without relation to anything external.”

Special relativity obliges us to abandon the absolute nature of time. We shall see, for example, that the time order of two events can depend on the relative motion of the observers who view them. One set of observers may find that a certain event *A* occurred before another event *B*, whereas according to a second set of observers, who are moving relative to the first, *B* occurred before *A*. This result is surely difficult to accept.

In some cases, a reversal of time ordering would be truly bizarre. Suppose that at event *A* a moth lands on the windshield of a moving car; the car clock reads 12:00. At event *B* another moth lands; the car clock now reads 12:05. For the driver of the car, the order of those events is a direct sensory experience: she can see both events happen right in front of her and can assert with confidence that *A* happened first. If observers on the ground were to claim that event *B* happened first, they would be denying

that sensory experience; moreover, the car clock would according to them be running backward! (It would read 12:05 before it reads 12:00.)

As we shall see, special relativity implies that moving clocks run slow. That is itself a strange result, but clocks running backward would be too much to swallow. No such disaster arises, however. In the case of the moths, event *A* happens first according to all observers. A reversal of time ordering can occur only for events spaced so far apart that no single observer (and no single clock) can be present at both. The order of such events is not a direct sensory experience for anyone; it can be determined only by comparing the readings of two *distinct* clocks, one present at event *A* and the other present at *B*. If two sets of observers disagree on the order of those events, no one's sensory experience is contradicted and no one sees any clock running backward. The proof of this assertion, given in chapter 5, depends on the fact that nothing can travel faster than light, one of the important consequences of special relativity.

A logical requirement of any theory is *causality*. If event *A* is the cause of event *B*, *A* must occur before *B*: the cause must precede the effect. We will see in chapter 5 that special relativity is consistent with the causality requirement. Whenever a cause-and-effect relation exists between two events, their time order is absolute: all observers agree on which one happened first.

Figure 1.1 shows a hypothetical experiment to illustrate the relativistic reversal of time ordering. Event *A* takes place in San Francisco and event *B* in New York. According to clocks at rest at those locations, *A* occurs before *B*. The same events are monitored by observers on spaceships moving from west to east at equal speeds; one ship is over San Francisco when event *A* occurs, and the other is over New York when event *B* occurs. Special relativity predicts that if the ships are moving fast enough, their clocks can show event *B* happening before *A*. Notice that no single clock is present at both events; the relevant times in the problem are recorded by four distinct clocks, two on the ground and two on the spaceships.

I hasten to add that no such experiment has ever been performed. The fastest available rockets travel a few kilometers per second, only about one hundred thousandth the speed of light. At that speed, the events of figure 1.1 would have to be separated in time by less than a millionth of a second if a reversal of time order were to be detectable. Moreover, the speeds of the two spaceships would have to be equal to within a very small tolerance. The experiment is just too hard to carry out. But we can be confident that if faster rockets were available and if other technical requirements were met, the effect could be detected.

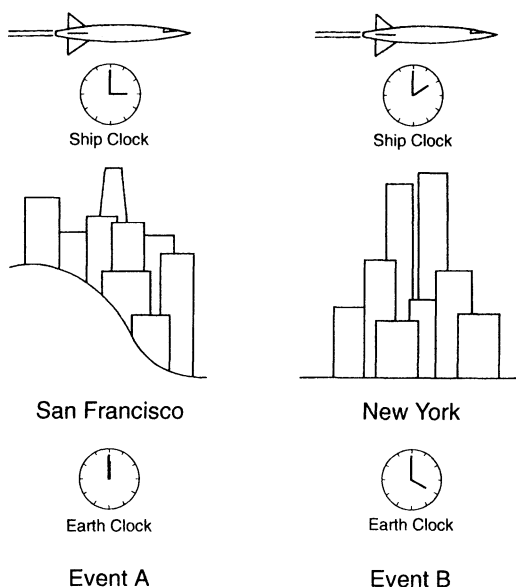


Fig. 1.1. Hypothetical experiment to demonstrate the reversal of time ordering predicted by special relativity. Event *A* occurs in San Francisco, event *B* in New York. Each event is detected by two sets of observers—one set fixed on earth and the other located on spaceships flying at equal (constant) speeds. Each set of observers measures the times of the two events on its own clocks, which have been previously synchronized. According to earth clocks, event *A* happens before *B*, whereas according to spaceship clocks, *B* happens before *A*. The time intervals shown on the clocks are much exaggerated.

The evidence that confirms special relativity comes principally from atomic and subatomic physics. In many experiments particles move at speeds close to that of light, and the effects of special relativity are dramatic. Particles are created and annihilated in accord with the famous Einstein relation $E = mc^2$. No understanding of such phenomena, or of the kinematics of high-energy particle reactions, would be possible without relativity. Thus Einstein's theory is confirmed daily in every high-energy physics laboratory. Particle reactions are not within the realm of everyday experience, however; in the latter realm, everything moves fairly slowly²

2. An obvious exception is light itself, which is part of everyday experience. Although any phenomenon that involves light is intrinsically relativistic, most opti-

and relativistic effects are not manifested. If the speed of light were much smaller, the effects of special relativity would be more prominent and our intuition concerning the nature of time would be quite different.

The preceding discussion is intended to provide a taste of what is to come and to encourage the reader to approach relativity with an open mind. I am not suggesting that any conclusion contrary to one's intuition be accepted uncritically, even though the context may be restricted to unfamiliar phenomena. On the contrary, any such conclusion must be vigorously challenged. Before abandoning ideas that appear to be self-evident, one must be satisfied that the experimental evidence is sound and the logical arguments are compelling.

1.2. EVENTS, OBSERVERS, AND FRAMES OF REFERENCE

I begin by defining some important terms. In relativity an *event* is any occurrence with which a definite time and a definite location are associated; it is an idealization in the sense that any actual event is bound to have a finite extent both in time and in space.

A *frame of reference* consists of an array of observers, all at rest relative to one another, stationed at regular intervals throughout space. A rectangular coordinate system moves with the observers, so that the x , y , and z coordinates of each observer are constant in time. The observers carry clocks that are synchronized: each clock has the same reading at the same time.³

Each observer records all events that occur at her location. Each event has four coordinates: three space coordinates and a time. By definition, the space coordinates are the coordinates of the observer who detected the event and the time of the event is the reading of her clock when it occurs.

A second frame of reference consists of another array of observers, all at rest relative to one another and all moving at the same velocity relative to the first set. They have their own coordinate system and their own (synchronized) clocks, and they also record the coordinates of events. The coordinates of a given event in two frames of reference are, in general,

cal phenomena can be explained without invoking the specific value of the speed of light. For example, refraction (the bending of a light ray when it crosses the boundary between air and glass) depends only on the ratio of the speeds of light in the two media. Hence a nonrelativistic theory of refraction is quite adequate.

Effects of relativity are manifested in experiments that depend on the time required for light to traverse a specified path, such as the Michelson-Morley experiment, discussed in detail in chapter 2.

3. The synchronization of clocks is discussed in detail in chapter 3.

different. The central problem of relativity is just to determine the relation between the two sets of coordinates; this turns out to be not so simple a matter as it first appears.

Throughout this book, whenever observations in two frames of reference are being compared, one frame will be called S and the other S' . Coordinates measured in frame S' will be designated by primed symbols, and those measured in frame S will be designated by unprimed symbols. Events will be labeled E_1, E_2, E_3 , and so on. Thus, x'_1, y'_1, z'_1 , and t'_1 denote the coordinates of event E_1 measured in frame S' ; x_2, y_2, z_2 , and t_2 denote the coordinates of event E_2 measured in frame S , and so on.

As an illustration, let us return to the problem of the child walking on a train. Figure 1.2 shows the child's motion as seen in two frames of reference, one fixed on the train (sketches [a] and [b]) and one fixed on the ground (sketches [c] and [d].) S is the ground frame and S' the train frame. The two sets of axes are parallel to one another. The train's motion as seen from the ground is taken to be in the x direction and the floor of the car is in the x - y plane. Since the child has no motion in the z direction, the figure has been simplified by omitting the z and z' axes.

In figure 1.2a, the child is just passing a train observer labeled H' ; this is event E_1 . The space coordinates of E_1 in S' are $x'_1=2, y'_1=1, z'_1=0$; its time coordinate t'_1 is the reading of the clock held by H' as the child passes her. Some time later, as shown in figure 1.2b, the child passes a second train observer, labeled J' ; this is event E_2 . The space coordinates of E_2 are $x'_2=3, y'_2=4, z'_2=0$; its time coordinate t'_2 is the reading of the clock held by J' .

Figure 1.2c shows event E_1 as seen in the ground frame. The child is just passing ground observer B . The space coordinates of E_1 in S are $x_1=2, y_1=1, z_1=0$; its time coordinate is read off B 's clock. Figures 1.2a and 1.2c should be thought of as being superposed: the positions of ground observer B , train observer H' , and the child all coincide when E_1 occurs.

Figure 1.2d similarly shows E_2 as seen in frame S ; the child is now passing ground observer Q . The space coordinates of E_2 in frame S are $x_2=5, y_2=4, z_2=0$. The positions of Q, J' , and the child all coincide at E_2 . Notice that B and H' , whose positions coincided at E_1 , no longer coincide at E_2 . As seen from the ground, all the train observers have moved to the right during the interval between the two events. (As seen from the train, all the ground observers have moved an equal distance to the left.)

Inspection of the figures reveals that the length of the child's path measured in the ground frame is greater than that measured in the train frame. The child's speed in the ground frame is correspondingly greater

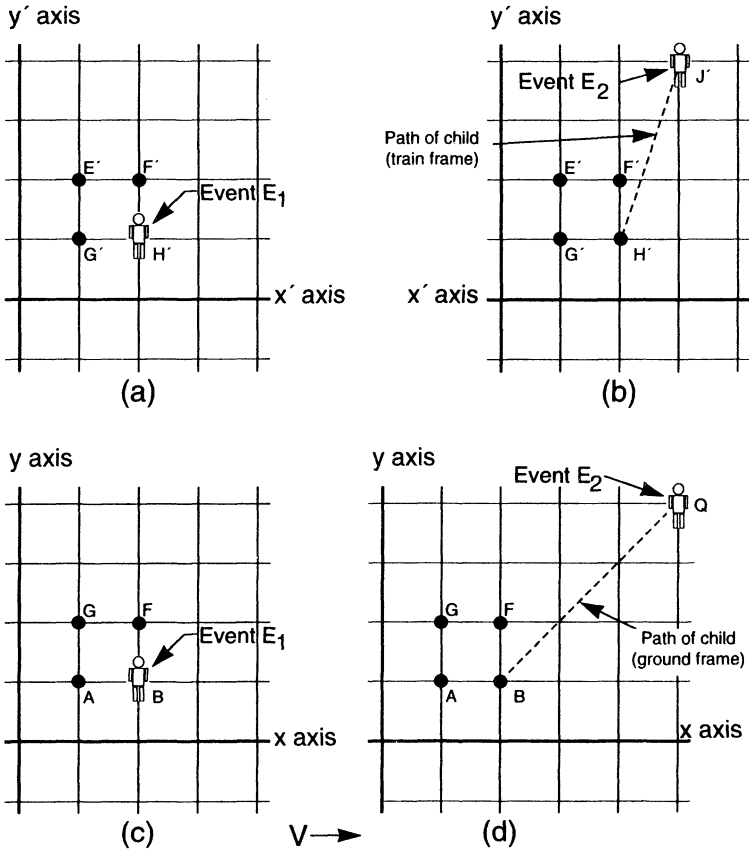


Fig. 1.2. Motion of child as seen in two frames of reference—one fixed on the train (primed coordinates, sketches [a] and [b]) and one fixed on the ground (unprimed coordinates, sketches [c] and [d]). (a) Child passes train observer H' (event E_1); (b) some time later, child passes train observer J' (event E_2). The path of the child, as seen in the train frame, is indicated by the dashed line in sketch (b). (c) Event E_1 is noted by ground observer B , whose location at that instant coincides with that of H' . (d) Event E_2 is noted by ground observer Q , who at that instant coincides with J' . The dashed line in sketch (d) shows the path of the child as seen in the ground frame.

(provided the elapsed time is the same in both frames, which is true in Galilean relativity).

The notion of a frame of reference as an (essentially infinite) array of observers is not intended to be a literal description of how measurements are carried out. It would be impractical, to say the least, to station observ-

ers throughout all space in the manner prescribed. But there is no reason in principle why that could not be done. In what follows, every event is assumed to be monitored by observers on the scene.

1.3. THE PRINCIPLE OF RELATIVITY AND INERTIAL FRAMES

The principle of relativity was first enunciated by Galileo in 1632. Galileo's argument is clear and graphically put.

Salviatus: Shut yourself up with some friend in the main cabin below decks on some large ship and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle which empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something toward your friend, you need throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though there is no doubt that when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than toward the prow, . . . despite the fact that during the time that you are in the air the floor under you will be going in a direction opposite to your jump. . . . Finally the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals by keeping themselves in the air. . . . The cause of all these correspondences of effects is the fact that the ships' motion is common to all the things contained in it.⁴

4. Galileo Galilei, *Dialogue Concerning the Two Chief World Systems*, translated by Stillman Drake (Berkeley and Los Angeles: University of California Press, 1953), 186–187. It is not clear whether Galileo ever actually performed the ship experiments. Following the quoted speech, Sagredo says, “Although it did not occur to me to put these observations to the test when I was voyaging, I am sure that they would take place in the ways you describe” (*Dialogue*, 188). This suggests that Galileo had not done the experiment. But see the remarks below.

Galileo is asserting, in effect, that *the laws of nature are the same in any two frames of reference that move uniformly with respect to one another*. If identical experiments are carried out by two sets of observers, with identical initial conditions, all the results will be the same. It follows that there is no way to determine by means of experiments carried out in a given frame of reference whether the frame is at rest or is moving uniformly. Only the relative velocity between frames can be measured. This set of assertions is called the principle of relativity.

Galileo's motivation was to refute Aristotle's argument that the earth must be standing still. If the earth were moving, Aristotle had claimed, a stone dropped from the top of a tower would not land at its base, since the earth would have moved while the stone was falling. Galileo argues that the earth plays a role entirely analogous to that of the ship in his example; just as a stone dropped from the top of a mast lands at its foot whether the ship is moving or at rest, so does one dropped from a tower on earth. And just as observations carried out within the ship cannot be used to decide whether the ship is standing still or moving uniformly, so the observed motion of objects on earth implies nothing about the motion of the earth other than that it is (approximately) uniform.

Although Galileo may not have carried out all the ship experiments, he definitely performed the falling rock experiment as well as many others on falling bodies. In a famous letter replying to Francesco Ingoli, who had attacked his views and sided with Aristotle, Galileo says, "whereas I have made the experiment, and even before that, natural reason had firmly persuaded me that the effect had to happen in the way that it indeed does."⁵

Several remarks are in order concerning Galileo's principle of relativity. First, the observations on which the principle was based were necessarily limited to quite slow speeds. Perhaps if the ship were moving very rapidly, shipborne observers might detect unusual effects that would enable them to conclude that their ship was indeed in motion. If that were to happen, the relativity principle would be only approximately valid. The laws of nature might be (very nearly) the same in two frames of reference

5. *The Galileo Affair*, editor and translator Maurice A. Finocchiaro (Berkeley and Los Angeles: University of California Press, 1989), 184. The motion of an object dropped from a moving vehicle had been debated long before Galileo. Tycho Brahe, as late as 1595, still sided with Aristotle, but Thomas Digges gave a correct analysis in his book, *A Perfit Description of the Celestial Orbes*, published in 1576. Giordano Bruno also studied the problem and came to the correct conclusion. According to Drake, Galileo probably knew about Bruno's work although he did not refer to it. (Bruno had been burned at the stake as a heretic.)

that move slowly relative to one another but quite different in two frames whose relative velocity is great. Galileo's observations obviously could not exclude such a possibility, and even today the direct evidence from physics in moving laboratories is limited to fairly low velocities. Indirect evidence, however, strongly supports the hypothesis that the relativity principle holds for any speed.

Galileo's experiments all deal with phenomena in what is nowadays called mechanics; on the basis of those experiments, therefore, one can conclude only that a principle of relativity applies to the laws of mechanics. Perhaps other experiments, involving different phenomena, can distinguish among frames.

Nineteenth-century physicists believed that electromagnetic and optical phenomena provide just such a distinction. According to the view prevalent during that period, there exists a unique frame of reference in which the laws of electromagnetism take a particularly simple form. If that were so, the principle of relativity would not apply to electromagnetic phenomena: the results of some experiments would depend on the observer's motion relative to the special frame.

Many experiments were performed with the aim of determining the earth's motion relative to the special frame, but they all failed to detect any effect of that assumed motion. The most important was the Michelson-Morley experiment, described in chapter 2.

For Einstein, it was aesthetically unsatisfying that a principle of relativity should hold for one set of phenomena (mechanics) but not for another (electromagnetism.) He postulated that Galileo's principle applies to *all* the laws of nature; this generalization forms the basis for special relativity.

The relativity principle has an important philosophical implication. If there is no way to distinguish between a state of rest and a state of uniform motion, absolute rest has no meaning. Observers in any frame are free to take their own frame as the standard of rest. Shore-based observers watching Galileo's ship are convinced that they are at rest and the ship is in motion, but observers on the ship are equally entitled to regard themselves as being at rest while the shore along with everything on it moves. The question, Which observers are *really* at rest? has no meaning if there is no conceivable experiment that could answer it. (According to observers in an airplane flying overhead, both shore observers and ship observers are in motion.)

In sum, the principle of relativity denies the possibility of absolute rest

(or of absolute motion). Motion can be defined only relative to a specific frame of reference, and among uniformly moving frames strict democracy prevails: any frame is just as good as any other. Any reference to a body “at rest” should be understood to mean “at rest in a frame of reference fixed on the earth” (or in some other specified frame).

The restriction to uniform motion is essential to the relativity principle. The laws of nature are *not* the same in all frames of reference.⁶ As Galileo fully realized, accelerated motion is readily distinguished from uniform motion. If a ship moves jerkily or changes direction abruptly, things behave strangely: suspended ropes do not hang vertically, a cake of ice placed on a level floor slides away for no apparent reason, and the flight pattern of Galileo’s butterflies appears quite different than it does when the ship is moving uniformly. Any of these effects tells the observers that their frame is accelerated.

The distinguishing feature of uniformly moving frames is that in any such frame the *law of inertia* holds: a body subject to no external forces remains at rest if initially at rest, or if initially in motion, it continues to move with constant speed in the same direction. In an accelerated frame, the law of inertia does not hold. Instead bodies seem to be subjected to peculiar forces for which no agent can be identified. Those forces, called “inertial forces,” have observable consequences.

Frames of reference in which the law of inertia holds are called *inertial frames*;⁷ all others are noninertial. In terms of this nomenclature, we can rephrase Galileo’s principle of relativity as follows:

If S is an inertial frame and S' is any other frame that moves uniformly with respect to S , then S' is also an inertial frame. All the laws of mechanics are the same in S' as in S , and no (mechanical) experiment can distinguish S' from S .

The discussion here will be confined almost entirely to inertial frames.

Observers in a given frame can determine whether their frame is inertial by carrying out experiments to test whether the law of inertia holds. A frame of reference fixed on earth satisfies the criterion fairly closely; for most purposes such a frame can be regarded as inertial. Because of the earth’s rotation, however, an earthbound frame is not strictly inertial.

Even a frame of reference fixed at the pole, which does not partake of the earth’s rotation, is not strictly inertial because the earth is moving in

6. See, however, the discussion of the principle of equivalence in chapter 8.

7. Einstein called them “Galilean frames.”

a curved orbit around the sun. And the sun is itself in orbit about the center of the galaxy. An inertial frame is an idealization in the sense that no experiment can assure us that our frame is strictly inertial, that is, that a body subject to no forces does not experience some tiny acceleration.

1.4. THE GALILEAN TRANSFORMATION

An event E occurs at time t at the point x, y, z , as measured in some inertial frame S . What are the coordinates (x', y', z', t') of E in another inertial frame, S' , that moves at velocity V relative to S ? The answer that any physicist would have given to this question before 1905 is the Galilean transformation, derived here. The derivation is straightforward and the results appear almost self-evident. As we shall see, however, special relativity gives a different answer.

For convenience, let the two sets of axes be parallel to one another, with their relative motion in the x (or x') direction (fig. 1.3). At some instant the origins O and O' coincide and all three pairs of axes are mo-

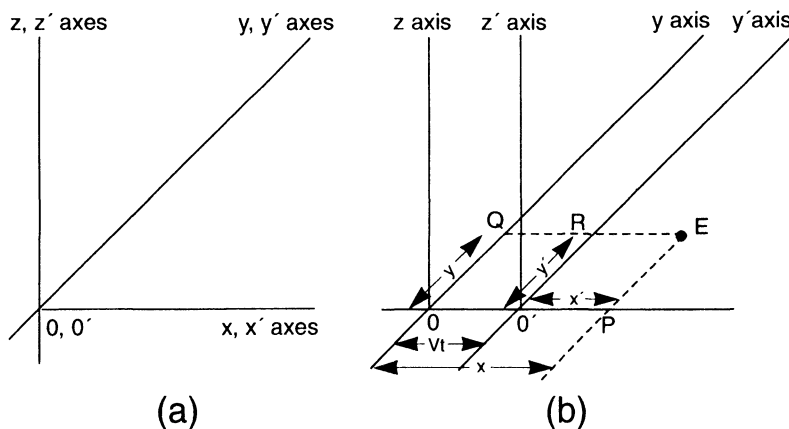


Fig. 1.3. Coordinates of an event in two frames of reference, according to Galilean relativity. The primed coordinate system is moving from left to right, as seen by observers in the unprimed system. (a) Primed and unprimed axes momentarily coincide. The clocks of all observers are arbitrarily set to zero at this instant. (b) The state of affairs at some later time, t . O' , the origin of the primed system, has moved a distance Vt down the x axis. The x and x' axes still coincide. The event in question occurs at the point labeled E . The space coordinates of the event in both frames are indicated. As the figure shows, y and y' are equal, but x' is less than x . (For simplicity, the z coordinate of the event is assumed to be zero.)

mentarily superposed (fig. 1.3a). At that moment⁸ all observers in both frames synchronize their clocks by setting all their readings to zero.

The relation between the times t and t' can be written directly. Since time is absolute, we have simply

$$t' = t \quad (1.1a)$$

The spatial coordinates of E in S' can be taken from the figure. At time t the two origins are separated by the distance Vt . Hence the relation between x and x' is

$$x' = x - Vt \quad (1.1b)$$

The y and z coordinates of E are the same in both frames:

$$y' = y \quad (1.1c)$$

$$z' = z \quad (1.1d)$$

Equations (1.1a–d) constitute the Galilean transformation.

Inverse Transformation

Suppose we are given the coordinates of an event in S' and want to find its coordinates in S . Solving equations (1.1a–d) for the unprimed coordinates in terms of the primed ones, we get

$$t = t' \quad (1.2a)$$

$$x = x' + Vt' \quad (1.2b)$$

$$y = y' \quad (1.2c)$$

$$z = z' \quad (1.2d)$$

which is the desired inverse transformation.

If primed and unprimed coordinates are interchanged and V is changed to $-V$, equations (1.1a–d) turn into equations (1.2a–d), and vice versa. This result is a logical necessity. It cannot matter which reference frame we choose to label S and which S' ; the same transformation law must apply. But V is defined as the velocity of S' relative to S . If we interchange the labels, the magnitude of the relative velocity is unchanged but its sign is reversed. (If ground observers see a train moving from left to right at a given speed, train observers must see the ground moving from right to left at the same speed.)

8. Because time is absolute in Galilean relativity, phrases like “at that moment” and “ t seconds later” have the same meaning in both frames. When we come to special relativity, we shall have to exercise great care in using such language.

Invariance of Distance

Suppose train and ground observers wish to measure the distance between two telephone poles situated alongside the track. The S positions of the poles are independent of time:

$$x_1 = A \quad x_2 = B \quad (1.3)$$

and the distance $x_2 - x_1$ between them is just $B - A$.

The equations of motion of the poles in S' are obtained by applying equation (1.2b) to both x_1 and x_2 in (1.3). The result is

$$x'_1 = A - Vt' \quad (1.4a)$$

$$x'_2 = B - Vt' \quad (1.4b)$$

Both these equations describe bodies moving from right to left at speed V , as they must.

Suppose train observers measure the position of pole #1 at time t'_1 and that of pole #2 at time t'_2 . The difference between the two readings is

$$x'_2 - x'_1 = B - A - V(t'_2 - t'_1) \quad (1.5)$$

Inasmuch as the poles are moving in S' , the two position measurements must be made at the same time if their difference is to yield the correct distance between the poles. With $t'_2 = t'_1$, equation (1.5) gives $x'_2 - x'_1 = B - A$, the same as the result obtained in frame S .

This discussion introduces the important concept of *invariance*. A quantity is said to be invariant if it has the same value in all frames of reference. I have shown that the spatial separation between two events that occur at the same time is invariant in Galilean relativity. The analogous statement in special relativity is not true.

Transformation of Velocity

Suppose a body moves in the x direction at velocity v , as measured in the ground frame S .⁹ If the body sets out from the origin at $t = 0$, its position at time t is

$$x = vt \quad (1.6)$$

Using equations (1.2a,b) to express x and t in terms of x' and t' , we obtain

9. Throughout this book, velocities of objects are denoted by lowercase letters. The velocity of one frame relative to another is denoted by a capital letter, usually V .

$$x' + Vt' = vt'$$

or

$$x' = (v - V)t' \quad (1.7)$$

Equation (1.7), like (1.6), expresses motion at constant velocity; the magnitude of the velocity, which we may call v' , is

$$v' = v - V \quad (1.8a)$$

The inverse transformation is obviously

$$v = v' + V \quad (1.8b)$$

With $v' = 1$ m/sec and $V = 30$ m/sec, equation (1.8b) gives $v = 31$ m/sec, the value cited earlier as the “commonsense” result.

If the motion is not confined to the x direction, we can write instead of equation (1.6)

$$x = v_x t \quad (1.9a)$$

$$y = v_y t \quad (1.9b)$$

$$z = v_z t \quad (1.9c)$$

where v_x , v_y , and v_z denote the three components of velocity in S .

When we transform to S' coordinates as before, the x equation reproduces the result expressed in equation (1.8a), with a subscript x on v and v' :

$$v'_x = v_x - V \quad (1.10a)$$

Since $y = y'$ and $t = t'$, equation (1.9b) becomes

$$y' = v_y t'$$

which implies that

$$v'_y = v_y \quad (1.10b)$$

Similarly, we find that

$$v'_z = v_z \quad (1.10c)$$

Only the component of velocity in the direction of the relative motion between frames changes when we change frames.

In deriving equation (1.8) we assumed that the velocity of the body in question was constant. If the velocity is changing, the result still holds provided v and v' refer to the *instantaneous* values of velocity (measured at the same time, of course). This is readily shown with the help of the

calculus; one simply differentiates equation (1.1) with respect to time. The same result can be derived by purely algebraic methods.

Combination of Galilean Transformations

Suppose two trains travel along the same track, one at velocity V and the other at velocity U relative to the ground. Let x' , y' , and z' and x'' , y'' , and z'' denote coordinates in frames of reference attached to the first and second train, respectively. The transformation from (x, y, z, t) to (x', y', z', t') is given by equation (1.1); that from (x, y, z, t) to (x'', y'', z'', t'') must be given by a similar set of equations, with U in place of V :

$$\begin{aligned}x'' &= x - Ut \\y'' &= y \quad z'' = z \\t'' &= t\end{aligned}\tag{1.11}$$

What about the transformation from the coordinates of the first train to those of the second? Eliminating (x, y, z, t) from equations (1.1) and (1.11), we find directly

$$\begin{aligned}x'' &= x' - (U - V)t' \\y'' &= y' \quad z'' = z' \\t'' &= t'\end{aligned}\tag{1.12}$$

Equation (1.12) describes another Galilean transformation, with relative velocity $U - V$. This is just the velocity of the second train as measured by observers on the first.

Acceleration

Finally, we examine the transformation properties of acceleration, the rate of change of velocity. This can be done without any equations.

According to equation (1.8), the velocities of a moving body in S and S' always differ by the same amount, V . If the velocity measured in S changes from v_1 to v_2 during some time interval, the velocity measured in S' changes from $v_1 - V$ to $v_2 - V$; the increment in velocity in S' is $v_2 - v_1$, the same as in S . Since acceleration is defined as change in velocity per unit time, it has the same value in both frames. Letting a and a' denote the accelerations measured in the two frames, we have simply

$$a' = a\tag{1.13}$$

Acceleration is invariant in Galilean relativity. We shall see in chapter 4 that in special relativity it transforms in a much more complicated manner.

1.5. STELLAR ABERRATION

An interesting application of the velocity transformation law is provided by stellar aberration, the change in the apparent direction of a star caused by the earth's motion around the sun.¹⁰ A similar effect can be detected when driving through a rainstorm: raindrops falling vertically appear to be moving obliquely.

Let S be a frame of reference in which the sun is at rest and the earth's orbit is in the $x-y$ plane. Suppose the orbital velocity V points in the x direction.

Consider a star that is located on the z axis and is not moving relative to the sun (fig. 1.4a). The analysis is simplest for this special case, although a similar result applies to any star.

The velocity components in frame S of a light ray that reaches earth from the star are

$$v_x = 0 \quad (1.14a)$$

$$v_y = 0 \quad (1.14b)$$

$$v_z = -c \quad (1.14c)$$

If the earth were not moving, a telescope pointed in the z direction would receive light from the star.

We want to find the direction of the light ray in S' , the earth's rest frame, which moves at velocity V relative to S . The Galilean velocity transformation, equation (1.10), gives the velocity components in S' :

$$v'_x = v_x - V = -V \quad (1.15a)$$

$$v'_y = v_y = 0 \quad (1.15b)$$

$$v'_z = v_z = -c \quad (1.15c)$$

Figure 1.4b shows the direction of the light ray in frame S' . The "apparent" direction of the star (the direction in which our telescope must be pointed) differs from its "true" direction by a small angle called the *aberration angle*, α . For the special case under consideration, the aberration angle is determined by the trigonometric relation

$$\tan \alpha = \frac{v'_x}{v'_z} = \frac{V}{c} \quad (1.16)$$

10. Aberration is not to be confused with stellar parallax, which is due to the changing *position* of the earth as it traverses its orbit. Unlike aberration, parallax depends on the distance of the star. Even for the nearest stars, the parallax angle is much smaller than the aberration angle.

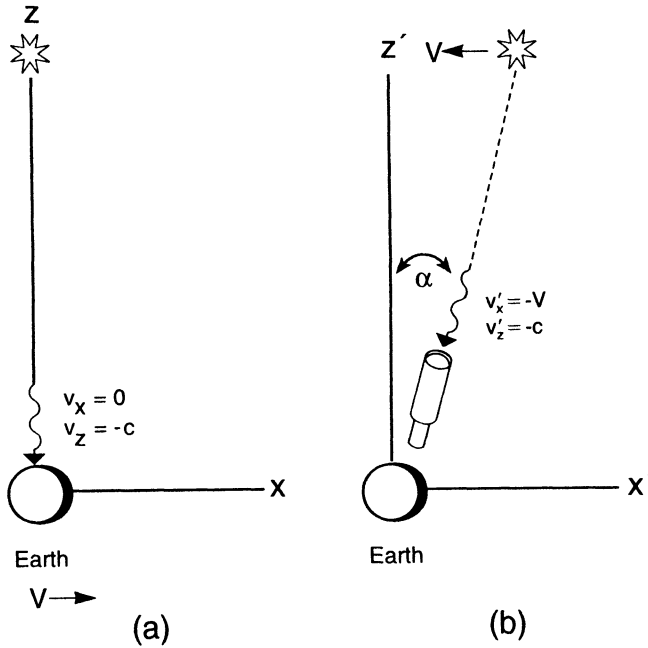


Fig. 1.4. Effect of the earth's orbital motion on the apparent position of a star. Sketch (a) is drawn in a frame of reference, S , in which the star is at rest on the z axis and the earth is moving in the x direction at velocity V . A light ray from the star, moving in the negative z direction, reaches earth. Sketch (b) shows the same ray in frame S' , in which the earth is at rest and the star is moving. S' moves at velocity V relative to S . The velocity components of the star are given by eq. (1.14) in S and by (1.15) in S' . To see the star, an astronomer must point his telescope at an angle, α , given by eq. (1.16); this effect is called aberration.

The value of V is known to be 30 km/sec. Equation (1.16) therefore gives $\tan \alpha = 10^{-4}$ or $\alpha = 20''$ of arc. Although this is a very small angle, it is readily measurable with a good telescope.

If the earth's motion were uniform, the aberration effect would be undetectable since the "true" direction of the star is unknown. But because the direction of the earth's orbital velocity changes regularly, the aberration effect likewise changes. Six months after the situation shown in the figure, the earth's velocity in frame S will have reversed its direction and in S' the star will appear to be on the other side of the z axis. Over the course of a year, the apparent position of the star traces out a circle whose radius is about 20 seconds of arc; for a star in an arbitrary direction, the

path is an ellipse. The effect was detected and explained correctly by James Bradley in 1725.¹¹

The aberration formula can also be derived by analyzing the situation from the outset in frame S . Between the time the light ray enters the telescope and the time it reaches the eyepiece, the telescope has moved. Unless the telescope is tilted, therefore, the light ray will run into the side of the instrument. The result obtained by such an argument is, of course, the same as equation (1.16). The advantage of the derivation given here is that it is readily generalized when we analyze the problem from the point of view of special relativity in chapter 4.

1.6. THE COVARIANCE OF PHYSICAL LAWS

The principle of relativity, set forth in section 1.3, asserts that the laws of physics are the same in any two inertial frames of reference. This principle can be rephrased as a requirement on the mathematical properties of physical laws.

A typical physical law expresses a mathematical relation between quantities like velocity, acceleration, and force. All these quantities must be measured in some frame of reference, say, S . Suppose we transform to a second frame S' using the Galilean transformation. If the primed quantities are related in exactly the same way as the corresponding unprimed ones, the relation is said to be *covariant* under the transformation. The principle of relativity demands that all the laws of mechanics be covariant under a Galilean transformation.

Consider the basic law of mechanics, Newton's second law:

$$F = ma \tag{1.17}$$

Here m is the mass of a body, F is the total external force acting on it, and a is its acceleration, all measured in some frame S .

Let F' and a' be the force and acceleration measured in some other inertial frame S' . If the second law is covariant, the relation

$$F' = ma' \tag{1.18}$$

must follow from (1.17).

To be perfectly general, we should have written equation (1.18) as

$$F' = m'a' \tag{1.19}$$

11. Bradley used his theory of aberration to deduce a quite accurate value for the speed of light.

with m' the mass appropriate to frame S' . In classical mechanics, however, mass is considered an intrinsic property of a body and must be invariant. Hence we can put $m' = m$.

We showed earlier (eq. [1.13]) that acceleration is invariant under a Galilean transformation: $a' = a$. Hence equation (1.18) follows from (1.17) if and only if the force F is also invariant,¹² that is, if

$$F' = F \quad (1.20)$$

One has to investigate case by case whether the forces that exist in nature satisfy the invariance requirement (eq. [1.20]). For example, the gravitational attraction between two bodies is given by Newton's law of gravity:

$$F = G \frac{m_1 m_2}{r^2} \quad (1.21)$$

Here m_1 and m_2 are the masses of the bodies, r is the distance between them, and G is a constant. We have already shown that distance is invariant under a Galilean transformation. Hence the gravitational force is indeed invariant and the law of gravity is consistent with the principle of relativity.

The most important example of a force law that is not covariant in Galilean relativity is electromagnetism. The laws of electricity and magnetism were codified in the late nineteenth century in a system known as Maxwell's equations. If one assumes that these equations hold in some frame S and makes a Galilean transformation to another frame S' , the equations in S' do not have the same form: Maxwell's equations are *not* covariant under a Galilean transformation. This was very troubling to Einstein and motivated his quest for a new theory.

There were three logical possibilities:

- (i) The relativity principle does not apply to electromagnetism; Maxwell's equations are valid only in one special frame of reference.
- (ii) The relativity principle does apply to electromagnetism, but Maxwell's equations are only approximately correct; they must be replaced by a more general set of equations that are strictly covariant.

12. If every term in a relation is invariant, as in the present example, the relation is obviously covariant. This is not a necessary condition, however. Covariance requires only that both sides of the equation transform in the same way. Suppose, for example, that under some hypothetical transformation, $a' = 2a$ and $F' = 2F$. Eq. (1.18) would be satisfied and the law would be covariant.

(iii) The relativity principle applies universally and Maxwell's equations are exact, but the Galilean transformation is wrong.

Alternative (i) was the one preferred by nineteenth-century physicists; the ether frame, discussed in chapter 2, was postulated to be the special frame in which Maxwell's equations hold. Einstein rejected that view and boldly asserted that alternative (iii), which a priori seems the least plausible, is in fact correct. This assumption led him to special relativity.

1.7. THE CONSERVATION OF MOMENTUM

As a final application of Galilean relativity, we examine the law of conservation of momentum and show that it is covariant.

Newton defined momentum (which he called "quantity of motion") as the product of mass and velocity. The modern symbol for momentum is p ; thus $p = mv$.

Momentum is a vector quantity: it has direction as well as magnitude. By definition, the momentum of a body points in the same direction as its velocity. The components of momentum along a given set of axes are

$$p_x = mv_x \quad p_y = mv_y \quad p_z = mv_z \quad (1.22)$$

Any of these components can be positive or negative.

An important law in classical mechanics is the conservation of momentum: the total momentum of an isolated system remains constant. The law is a consequence of Newton's second and third laws.

The most common application of the conservation of momentum is to problems that involve collisions. When two bodies collide, the momentum of each body changes as a consequence of the forces exerted on it by the other. The *total* momentum of the system, however, is the same after the collision as before. If the collision involves motion in more than one dimension, each component of momentum is separately conserved.

Figure 1.5 illustrates the conservation of momentum in collisions between bodies of equal mass m . In each case body A , moving at velocity v in the x direction, collides with body B , which is initially at rest. The total momentum of the system before the collision is mv . In collision (a), body A comes to rest and B moves at velocity v after the collision in the direction of A 's initial motion. The final momentum is mv , in accord with the conservation law.

Figure 1.5b shows a different possible outcome: A and B stick together, and both move at velocity $v/2$ after the collision. The final momentum of each body is $mv/2$, and the total is again mv ; momentum is conserved.

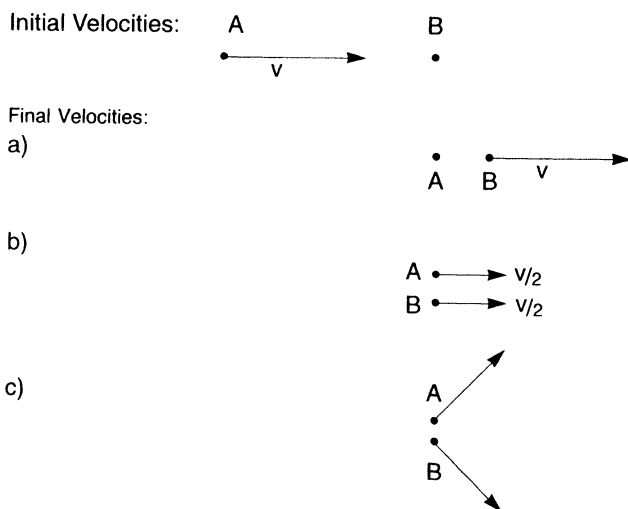


Fig. 1.5. The conservation of momentum according to Galilean relativity. Shown are three collisions, in each of which momentum is conserved. The initial state in each collision is that shown in the top sketch: body *A*, moving in the *x* direction, collides with body *B*, initially at rest. The bodies have equal masses. After collision (a), *A* is at rest and *B* moves with *A*'s original velocity, *v*. After collision (b), each body moves with velocity *v*/2 in the *x* direction. After collision (c), each body has some *y* motion as well as some *x* motion. The *y* components of velocity (and of momentum) are equal and opposite; thus the total *y* momentum is zero.

Figure 1.5c shows yet another possible outcome. In this case, each body has some *y* velocity and therefore some *y* momentum after the collision. Since the initial *y* momentum was zero, the total *y* momentum after the collision must also be zero. The *y* momenta of the emerging bodies must have equal magnitudes and opposite signs; since the masses are equal, the *y* velocities must likewise be equal and opposite.

As these examples demonstrate, conservation of momentum does not determine the outcome of a collision. If we know only the initial velocities of the colliding bodies, we cannot predict whether the final velocities are those shown in (a), in (b), in (c), or something different still.¹³ The outcome depends on data that have not been specified, such as the elastic

13. The conservation law (see eq. [1.23]) is one equation with two unknowns (the two final velocities). It has an infinite number of solutions.

properties of the bodies and whether the collision is head-on or at a glancing angle. A head-on collision between two billiard balls would lead to outcome (a), whereas a glancing collision between the same billiard balls could lead to outcome (c). Colliding lumps of putty are likely to stick together, outcome (b).

Consider a collision between two bodies, A and B , whose motion is confined to one dimension. Conservation of momentum is expressed by the relation

$$m_A v_A + m_B v_B = m_C v_C + m_D v_D \quad (1.23)$$

where C and D refer to the bodies that emerge from the collision. C and D might be the same as A and B , but they might be different. Such “re-arrangement” collisions are of particular interest in nuclear physics.

To prove that momentum conservation is a covariant law, we assume that equation (1.23) holds in some inertial frame S and show that the same relation holds also in any other inertial frame S' .

The velocity of each body in S' is related to its velocity in S by the Galilean velocity transformation, equation (1.7):

$$v_A = v'_A + V, \quad v_B = v'_B + V, \quad \text{and so on} \quad (1.24)$$

where, as usual, V is the speed of frame S' relative to S .

Using equation (1.24) we can express each term in equation (1.23) in terms of velocities measured in S' . The result is

$$m_A(v'_A + V) + m_B(v'_B + V) = m_C(v'_C + V) + m_D(v'_D + V) \quad (1.25)$$

Expanding and grouping terms, we obtain

$$m_A v'_A + m_B v'_B = m_C v'_C + m_D v'_D + V(m_C + m_D - m_A - m_B) \quad (1.26)$$

Equation (1.26) represents momentum conservation in frame S' , provided the last term on the right side vanishes. Since V is not zero, this requires that

$$m_C + m_D = m_A + m_B \quad (1.27)$$

Condition (1.27), which must be satisfied if momentum conservation is to be a covariant law, expresses the conservation of mass. In classical mechanics, mass can be neither created nor destroyed; equation (1.27) must be valid. If, as a result of a collision, the colliding bodies exchange mass or even break up into many fragments, the total mass of the emerging bodies must be exactly the same as the total mass of the bodies that collided. As we shall see, that statement is not true in special relativity.

The proof of covariance is readily extended to collisions in more than one dimension. Each component of momentum can be treated separately. The preceding analysis shows that if x momentum is conserved in S , it is conserved as well in S' . (V , the relative velocity of the two frames, is assumed to be in the x direction.) But according to equation (1.10), the y velocity of each body is the same in S' as in S . Since momentum is velocity times mass, it follows that the y momentum of each body is likewise the same in both frames. Hence if y momentum is conserved in frame S , it is likewise conserved in S' . The same is true of z momentum.

In summary, the law of conservation of momentum, with momentum defined as mass times velocity, is covariant under a Galilean transformation, provided that mass is conserved. We shall see in chapter 7 that in special relativity, momentum must be redefined if the conservation law is to be covariant.

PROBLEMS

1.1. A train moves at a constant speed. A stone on the train is released from rest.

(a) Using the principle of relativity, describe the motion of the stone as seen by observers on the train.

(b) Using the Galilean transformation, describe the motion of the stone as seen by observers on the ground. Draw a sketch.

1.2. This problem deals quantitatively with the experiment of problem 1.1. Let S denote the ground frame of reference and S' the train's rest frame. Let the speed of the train, as measured by ground observers, be 30 m/sec in the x direction, and suppose the stone is released at $t' = 0$ at the point $x' = y' = 0$, $z' = 7.2$ m.

(a) Write the equations that describe the stone's motion in frame S' . That is, give x' , y' , and z' as functions of t' . (Note: A body starting from rest and moving with constant acceleration g travels a distance $\frac{1}{2}gt^2$ in time t . Gravity produces a constant acceleration whose magnitude is approximately 10 m/sec/sec.)

(b) Use the Galilean transformation to write the equations that describe the position of the stone in frame S . Plot the stone's position at intervals of 0.2 sec, and sketch the curve that describes its trajectory in frame S . What curve is this?

(c) The velocity acquired by a body starting from rest with acceleration g is gt . Write the equations that describe the three components of the stone's velocity in S' , and use the Galilean velocity transformation to find the velocity components in S .

(d) Find the magnitude of the stone's speed at $t = 1$ sec in each frame.

1.3. A jetliner has an air speed of 500 mph. A 200-mph wind is blowing from west to east.

(a) The pilot heads due north. In what direction does the plane fly, and what is its ground speed? (Hint: Define a frame of reference S' that moves with the

wind. In S' there is no wind; hence the plane always moves in the direction it is headed, at 500 mph. Use the velocity transformation to find the components of the plane's velocity in the ground frame.)

(b) In what direction should the pilot head in order to fly due north? What is the plane's ground speed in this case?

1.4. A river is 20 m wide; a 1 m/sec current flows downstream. Two swimmers, A and B , arrange a race. A is to swim to a point 20 m downstream and back while B swims straight across the river and back. Each can swim at 2 m/sec in still water.

(a) In what direction should B head in order to swim straight across? Illustrate with a sketch. (See the hint for the preceding problem.)

(b) Who wins the race, and by how much time?

1.5. An elastic collision is one in which kinetic energy as well as momentum is conserved, that is, the total kinetic energy after the collision is equal to the total initial kinetic energy. The Newtonian definition of kinetic energy is $K = \frac{1}{2} mv^2$.

Consider the collision shown in fig. 1.5a. The mass of each body is 2 kg; the initial velocity of body A is 0.6 m/sec. In frame S , the frame in which the figure is drawn, the collision is obviously elastic. (The initial kinetic energy of B and the final kinetic energy of A are both zero; in the collision A 's momentum and kinetic energy are simply transferred to B .)

Analyze the same collision in a frame S' that moves to the right at 0.2 m/sec relative to S . Find the kinetic energy of each body before and after the collision and verify that the collision as seen in S' is elastic.

1.6. The object of this problem is to investigate whether, as suggested by the result of problem 1.5, the definition of an elastic collision is invariant under a Galilean transformation.

Consider the general one-dimensional collision discussed in section 1.7, in which bodies A and B collide and bodies C and D emerge. (C and D might be the same as A and B , or they might be different.) Assume that momentum is conserved, that is, eq. (1.23) is satisfied.

(a) Write the equation that expresses the conservation of kinetic energy in frame S . Now transform to a frame S' that moves at velocity V relative to S . Show that kinetic energy is conserved in S' , provided mass conservation is satisfied.

(b) Suppose the collision is inelastic: the total kinetic energy of C and D in frame S differs from the total kinetic energy of A and B by an amount Q . Is the value of Q invariant? Justify your answer.

1.7. Raindrops are falling vertically at 2 m/sec. A person is running horizontally at 3 m/sec. At what angle to the vertical should she hold her umbrella for maximum effectiveness? (Consider the path of the raindrops in the runner's rest frame.)

1.8. Analyze the stellar aberration effect for a star that lies in the plane of the earth's orbit. How does the magnitude of the aberration angle vary as the earth traverses its orbit?